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LETTER TO THE EDITOR

Lacunarity and percolation phase transitions

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Abstract. The lacunarity is computed for randomly diluted two-dimensional lattices in terms of the chemical distance. The crossover behaviour and the question of universality are discussed.

Fractal lattices are not translationally invariant and the mass distribution around each occupied site is different from site to site. The fractal dimension characterizes only the average of the mass distribution; the fluctuation around the average is characterized by the lacunarity which may be defined as the relative mean-square width of the distribution. The lacunarity has been studied in thermal phase transitions as a parameter which may prove relevant for the Ising critical behaviour in deterministic fractals [1-8]. The purpose of this letter is to study the lacunarity with random fractals.

Random fractals do indeed show remarkable fluctuation phenomena (for a review, see [9]). Some time ago, Coniglio *et al* [10] asked the question: when large percolation clusters are simulated on a computer many times, does the cluster number N_s fluctuate from sample to sample in the 'usual' way like the particle number in an ideal gas for which the compressibility never diverges, or does it fluctuate in the 'critical' way like the particle number in a real gas for which the compressibility diverges near the critical point? The fluctuation turns out to be equal to the square root of the average cluster number, as in the 'usual' way. But the average cluster number itself sharply peaks only near the percolation threshold, as in the 'critical' way. Stauffer [11] and Coniglio and Stauffer [12] studied the fluctuation of S_{max}, the number of lattice sites of the largest cluster, and Kapitulnik *et al* [13] studied the fluctuation of $M(L)$, the number of sites within a volume of L^d that surrounds an occupied site of the spanning cluster. Both quantities fluctuate in the 'critical' way; the fluctuation of $M(L)$ is the same as, not the square root of, the average of $M(L)$. Moreover, when these quantities are divided by L^d, the results scale with L like the order parameter and therefore the fluctuation of $M(L)$ is the percolation analogue of the diverging compressibility (or the magnetic susceptibility) [11-14]. In terms of lacunarity, all of this can be summarized by saying that the lacunarity is scale independent. This suggests that the local details are irrelevant, and we will argue shortly that the lacunarity is indeed universal for all lattice structures.

The mass fluctuation and the consequent lacunarity are computed in this report for randomly diluted two-dimensional lattices using the chemical distance (e.g. see [15, 16]) rather than the Pythagorean distance. Our original purpose in using the chemical distance was to take into account the fact that the percolation takes place only through the connected pores. This turns out to be not crucial, but it saves substantial computer time. Thus we define the mass at a chosen site as the total number of active

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sites which can be reached (from the chosen site) within the chemical distance of n, or $M(i, n)$, where the first argument represents the lattice site. Since $M(i, n)$ is different from site to site, an average is taken over all lattice sites to obtain $\langle M(n)\rangle$. The chemical dimension D is defined by the average asymptotic behaviour, $\langle M(n)\rangle \sim n^D$. The lacunarity is then defined as the relative root mean-square width of the distribution, i.e.

$$L(n) = \Delta M(n)/\langle M(n)\rangle \tag{1}$$

where

$$\Delta M(n) = (\langle M(n)^2\rangle - \langle M(n)\rangle^2)^{1/2}. \tag{2}$$

We consider four different two-dimensional lattice structures; square, triangular, hexagonal and Kagome. The initial lattice, which contains approximately $N = 28\,900$ active sites, is properly wrapped around to satisfy the periodic boundary conditions. A randomly chosen group of N_i sites is then turned into inactive (impure) sites. Consider first the cases of $p = (N - N_i)/N < p_c$ (the percolation threshold). For p slightly less than p_c, $L(n)$ begins with a small value and stays with very little change until n reaches a 'turning point' where it begins to fall off, as shown in figure 1. As p is decreased further, the turning point is pushed further out, but the fall-off follows the same power law, $L(n) \sim n^{-\rho}$. We find $\rho = 1.1 \pm 0.1$, which is reasonably close to the exact value $\rho = d/2 = 1$ (see below). Finally when the percolation threshold is reached, the lacunarity of the percolating cluster does not fall off any more; it maintains a constant value for all values of n, as shown in figure 2.

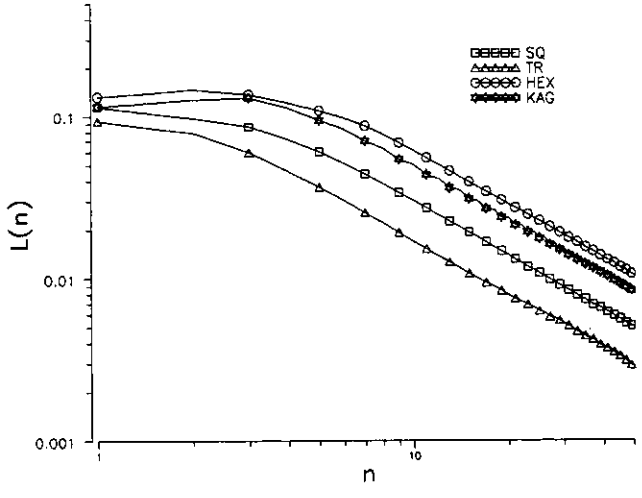

Figure 1. Lacunarity for $p = 0.95 > p_c$.

The turning point is marked by the percolation correlation length ξ, and the above crossover behaviour may be summarized by:

$$L(n, \xi) = \Lambda(n/\xi) \tag{3}$$

where $\Lambda(x) \sim$ constant for $n \ll \xi$, and $\Lambda(x) \sim x^{-d/2}$ for $n \gg \xi$. It is interesting to note that the degree of homogeneity of the scaling crossover function is zero. This gives more meaning to the notion of self-similarity which exists in the region $n \ll \xi$ and may be manifested by the real space renormalization [17, 18]. The order parameter is invariant under the renormalization which shrinks the length scale and lowers the

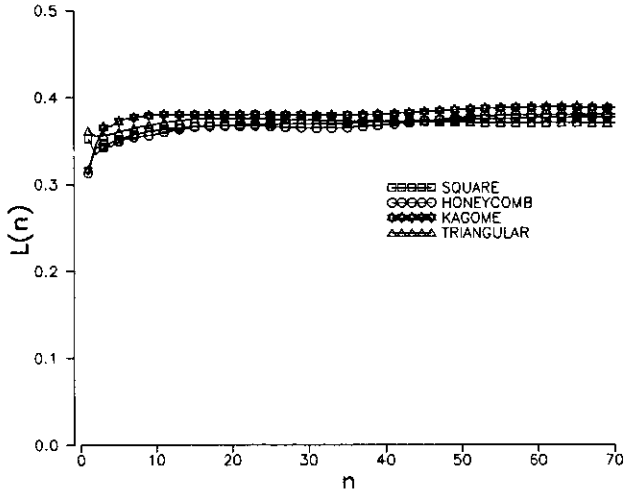

Figure 2. Lacunarity of percolating clusters.

'contrast'. What about the fluctuation and the lacunarity? The fluctuation is variant, but the lacunarity is invariant, under the renormalization.

The region $n < \xi$ is the fractal region [14], in which both $M(n)$ and $\Delta M(n)$ follow the same power law, n^D, and the region $n > \xi$ is the homogeneous region (14), in which $M(n) \sim n^d$, and $\Delta M(n) \sim n^{d/2}$. The crossover behaviour of $M(n)$ and $\Delta M(n)$ are given by [9, 13, 14]:

$$M(n, \xi) = n^D m(n/\xi) \tag{4}$$

where $m(x) \sim$ constant for $n < \xi$, and $m(x) \sim x^{d-D}$ for $n > \xi$; and

$$\Delta M(n, \xi) = n^D \mu(n/\xi) \tag{5}$$

where $\mu(x) \sim$ constant, for $n < \xi$, and $\mu(x) \sim x^{d/2-D}$, for $n > \xi$. The fact that the mass fluctuation is not 'critical' in the homogeneous region was reflected in the fall-off exponent of the lacunarity $\rho = d/2$. For the chemical dimension, we find $D = 1.6 \pm 0.1$ from the results shown in figure 3; a more accurate value is $D = D_f/d_m = 1.89/1.13 = 1.68$, where D_f is the fractal dimension (21), and d_m is the minimum dimension [19, 20].

The chemical distance in (1) may be converted into the Pythagorean distance in the following way. Consider a percolating cluster on a large lattice $N \times N$. For a given value of $n < N$, one may assume that $M(i, n)$ is the mass of a finite spanning cluster which spans on a *finite lattice of linear size n*. The linear size is equal to n because, for a given value of chemical distance n, the maximum possible Pythagorean distance is n. Corresponding to each lattice site i, there is a 'spanning' cluster whose mass is $M(i, n)$. Thus, by sweeping through all occupied lattice sites i, one collects an ensemble of 'spanning' clusters. Our site average is the same as the usual ensemble average. Following Stauffer [11], we may treat

$$F(n) = (\langle M(n)^2 \rangle - \langle M(n) \rangle^2)/n^d \tag{6}$$

as a scale-dependent percolation analogue of the magnetic susceptibility. The lacunarity may then be written entirely in terms of normalized percolation quantities as

$$L(n) = n^{-d/2} F(n)^{1/2} / \langle m(n) \rangle \tag{7}$$

where $\langle m(n) \rangle = \langle M(n) \rangle / n^d$ is the normalized order parameter.

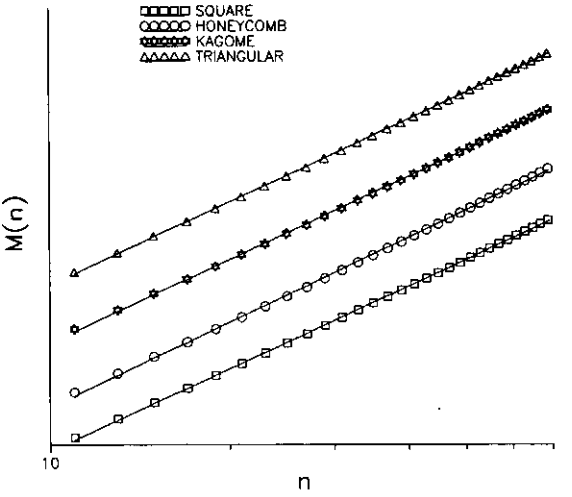

Figure 3. Plot of $\langle M(n)\rangle$ for percolating clusters. The connecting lines represent the best power-law fit. The logarithmic scale of the vertical axis is arbitrary; each curve has been arbitrarily shifted to avoid overcrowding.

Written in this form, it is easy to see the scale-independence and the universality of the lacunarity. According to the finite-size scaling theory [21], $\langle m(n)\rangle \sim n^{-\beta/\nu}$, and $F(n) \sim n^{\gamma/\nu}$. The percolation hyperscaling law, $\gamma + 2\beta = d\nu$, then leads to the scale-independent behaviour. Invoking the finite-size scaling ansatz of Privman and Fisher [22, 23] with the 'ghost field' [24] serving the role of magnetic field, one can also easily see that the system-dependent metric which appears in the numerator and the denominator of (7) are cancelled out leaving only system-independent derivatives of the universal scaling function. It seems fair to regard the results of figure 2 as bearing out this prediction.

As shown by (7), the lacunarity is a cumulant ratio for the percolation order parameter. Similar cumulant ratios of higher orders have played important roles for thermal phase transitions (see, for example, Binder in [23]). Binder's fourth-order cumulant is scale independent at the critical temperature, as is the lacunarity at the percolation threshold.

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